Random hyperbolic 3-manifolds

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Talk workshopped with Khaterine Merkl, Bratati Som and Leyla Yardimci

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Motivation

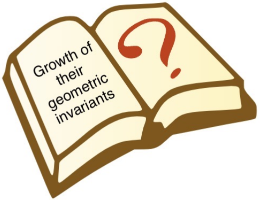
For 2-manifolds...

For 3-manifolds...

Thurston's geometrization conjecture

$$\mathbb{S}^3,\ \mathbb{E}^3,\ \mathbb{H}^3,\ \mathbb{H}^2\times\mathbb{R},\ \mathbb{S}^2\times\mathbb{R},\ \tilde{SL}(2,\mathbb{Z}),\ \mathrm{Nil},\ \mathrm{Sol}$$

$\hookrightarrow$



One question to ask

Given a random hyperbolic 3-manifold...

of large volume...

what is the probability that it has 2000 closed geodesics of length at most 10?

Random manifold

$$\{\text{ Set of manifolds }\} + \{\text{ Probability measure }\} = (\Omega, \mathbb{P})$$

$$\Downarrow$$

$\Rightarrow$ What is the probability that a random manifold has a certain property?

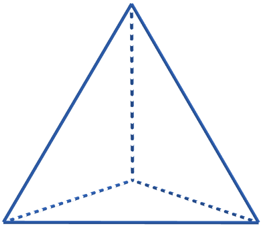
CONSTRUCTING RANDOM 3-MANIFOLDS

Random triangulation

Introduced by Bram Petri and Jean Raimbault (2020).

General idea: To construct manifolds by randomly gluing polyhedra together along their faces.

1st attempt:



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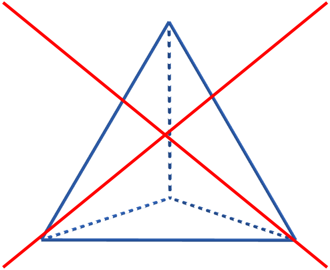
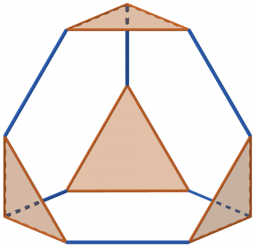


This doesn't work!

The neighbourhoods of the vertices are not typically homeomorphic to $\mathbb{R}^3$.

The model M_N

Solution:

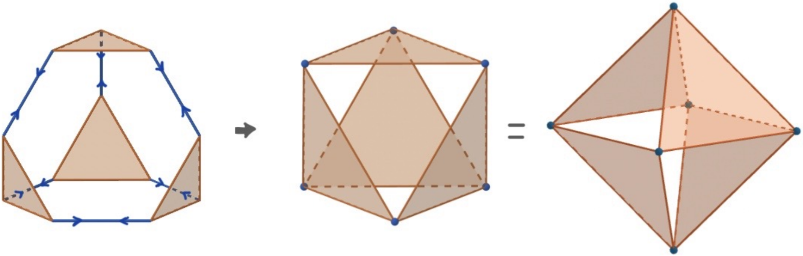


By gluing them along their hexagonal faces, we obtain:

$\Rightarrow$ A compact 3-manifold with boundary M_N,

where N is the number of tetrahedra.

The model Y_N

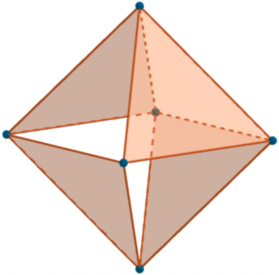


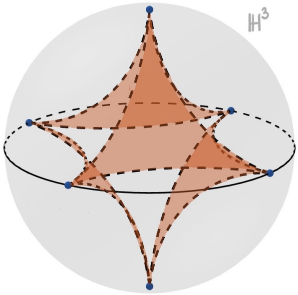
If we do this transformation for every tetrahedra in M_N, we obtain:

$\Rightarrow$ A new 3-manifold with boundary Y_N, made of octahedra.

The model Y_N: Hyperbolic metric

We endow each octahedron in Y_N with the hyperbolic metric of an ideal right-angled octahedron in $\mathbb{H}^3$.





▻ With this, Y_N becomes a complete finite volume hyperbolic 3-manifold with totally geodesic boundary.

Our question

Given a ~~random hyperbolic 3-manifold~~

~~of large volume~~

Y_N

what is the probability that it has 2000 closed geodesics of length at most 10?

COUNTING CLOSED GEODESICS

The lenght spectrum

Definition

The length spectrum $L(M)$ of a hyperbolic manifold M is the set of lengths of closed geodesics in M.

▻ We encode $L(Y_N)$ through the function:

$$L \longrightarrow C_L(Y_N) = \#\{\text{closed geodesics of length} \leqslant L \text{ on } Y_N\},$$

where $L > 0$ and $C_L(Y_N)$ is a random variable.

Poisson distribution

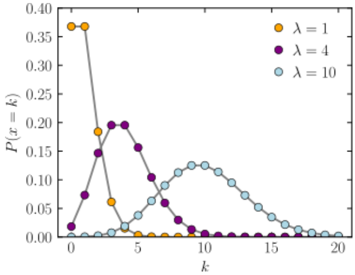
$\mathbb{P}$[there are k events happening in a specified interval [0, L]]*

$*$ provided that they are independent and occur with a known constant mean rate.

Definition

A random variable $Z : \Omega \to \mathbb{N}$ follows a *Poisson distribution of parameter* $\lambda > 0$ if for any $k \in \mathbb{N}$,

$$P[Z = k] = \frac{\lambda^k e^{-\lambda}}{k!}.$$



Main result

Theorem (Roig Sanchis)

As $N \to \infty$, $C_L(Y_N)$ converges in distribution to a Poisson random variable $C_L(Y)$ with explicit parameter $\lambda(L)$.

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THANK YOU!